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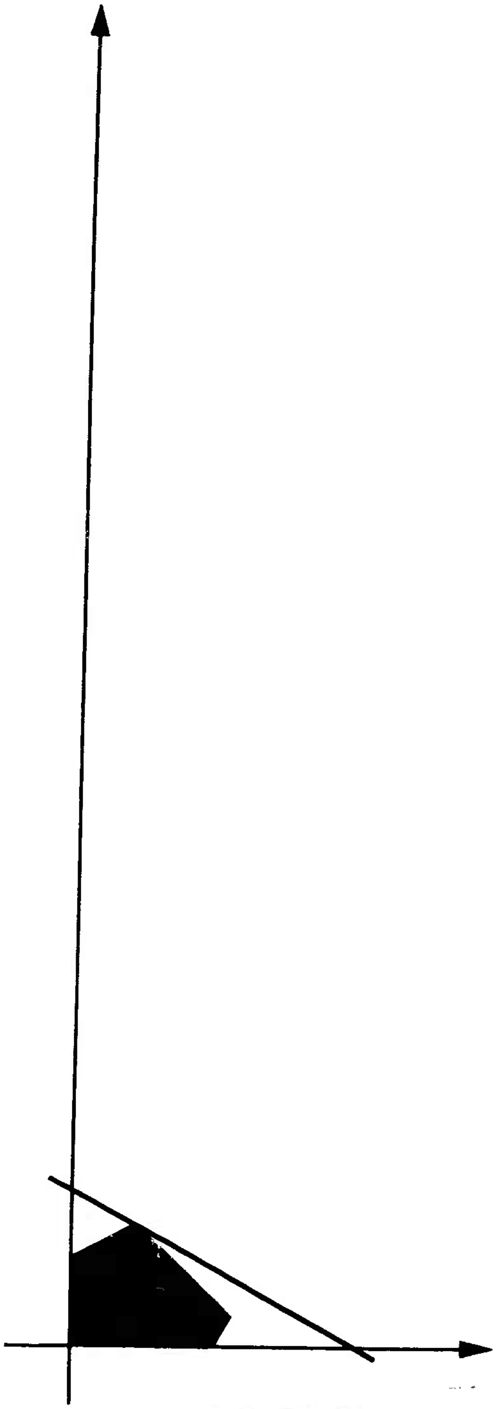

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Robert M. Oliver
Operations Research Center
University of California, Berkeley

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ABSTRACT

The author uses two equations to relate velocity, density, flow rate, and travel time in a traffic stream. One is the familiar equation of continuity, the second is an integral equation expressing the conservation of fluid in terms of travel times in the fluid. A situation is investigated where time-dependent flow rates into a bottleneck temporarily exceed its capacity. Expressions are found for queue sizes, the location and velocity of shock waves, and delays to travellers in the stream.

TRAVEL TIMES THROUGH SHOCK WAVES

1. The Conservation of Fluid

In recent years Lighthill and Whitham (1955), Richards (1956), De (1956), Bick and Newell (1960) and Pipes (1961) have discussed the propagation of shock waves in a traffic stream. It is assumed that the vehicles can be replaced by a continuous, compressible fluid. It is then shown that the flow rate, $q(x,t)$ and the density, $k(x,t)$, satisfy the familiar equation of continuity,

$$\frac{\partial q}{\partial x} + \frac{\partial k}{\partial t} = 0 \quad . \tag{1}$$

The dimensions of $q(x,t)$ are vehicles per unit of time and the dimensions of $k(x,t)$ are vehicles per unit distance. In high-density regions it is usually assumed that the velocity, $v(x,t)$ is only a function of the density; hence the flow rate,

$$q(x,t) = k(x,t)v(x,t) \tag{2}$$

need only be written as a function of the density. The derivation of (1) follows from the fundamental law that the traffic fluid is conserved at all points along the road while special cases of (2) have been conjectured on theoretical as well as experimental grounds. In the sections which follow we assume that $q(x,t)$ in (2) has a well-defined but not necessarily unique inverse.

Solutions of (1) and (2) satisfying appropriate boundary conditions at time zero have been found for certain flow patterns in the traffic stream. Lighthill and Whitham (1956) discuss the qualitative characteristics of the propagation of shock waves through the fluid. Richards (1956) assumes the linear Greenshields (1945) relation between velocity and density in (2) and from (1)

finds the rate at which discontinuities in the stream density (i.e., shock waves) propagate away from an intersection. De (1956) finds explicit solutions for the propagation of shock waves through a bottleneck whose capacity falls off linearly with time, reaches a minimum value and then increases linearly with time to its normal value. In each of these papers the authors assume that the density-flow relations of (2) which have been found to hold under steady-state flows are also valid for the time-dependent situations. Bick and Newell (1960) extend these early results by studying the interaction between two opposing streams of traffic when the velocity in one stream depends on the density in the opposing as well as its own stream.

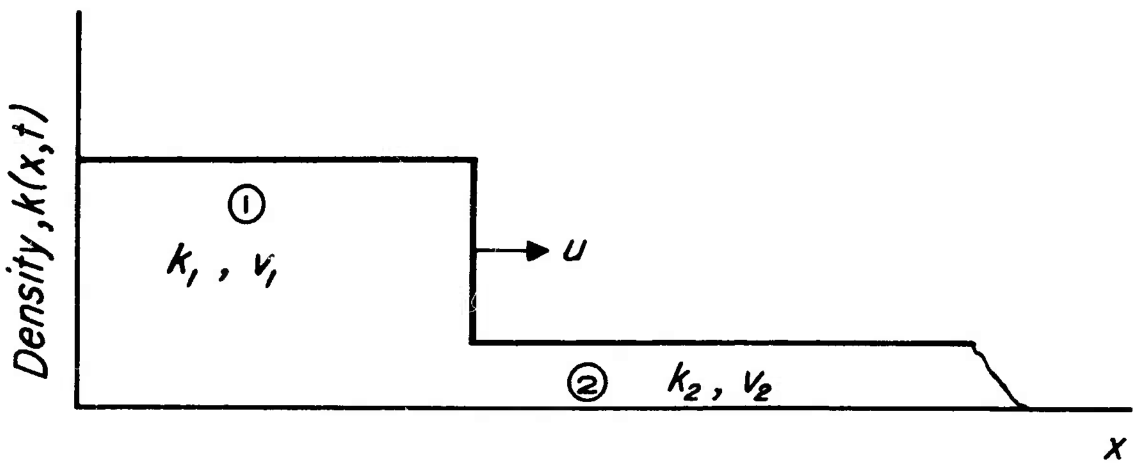

Fig. 1 – THE MOTION OF A SHOCK WAVE.

A lucid derivation of the formation and motion of a shock wave has been given by Pipes (1961); in this paper he also points out that the Lighthill-Whitham and Richards theory of shock waves are equivalent. Consider the low (1) and the high (2) density regions in Figure 1. The interface separating these two regions is a shock front which travels with a velocity, u . The speed of the stream

in region (1) relative to the shock wave is $v_1 - u$; the relative speed in region (2) is $v_2 - u$. Since the net flow rate of vehicles into the wave must equal the net flow rate out of the wave, we have

$$k_1(v_1 - u) = k_2(v_2 - u) \quad . \tag{3}$$

The velocity of the shock wave is therefore equal to the ratio of the differences in flow to the differences in density in each region.

$$u = \frac{v_2 k_2 - v_1 k_1}{k_2 - k_1} = \frac{q_2 - q_1}{k_2 - k_1} \tag{4}$$

In the limiting case of differential arguments where k and q are continuously varying quantities we have $u = \Delta q/\Delta k \rightarrow dq/dk$. If we consider the special case where q_2 is zero but where k_2 is large and k_1 is negligible, we find that the shock wave moves backwards (negative velocity) with a velocity equal to the incoming flow rate divided by the density in the stalled region. In other words, the inventory of vehicles which is increasing at a rate q_1 is stored in a space equal to the number of vehicles times the average spece, k_2^{-1} , occupied by each vehicle. The velocity of the shock wave simply corresponds to the rate at which inventory space must be provided to accomodate vehicles. If the flow rate can be written solely in terms of the density, Equation (1) can be rewritten in the form,

$$u(k) \frac{\partial k}{\partial x} + \frac{\partial k}{\partial t} = 0 \quad . \tag{5}$$

Pipes has shown that (1) can be written in terms of flow rates by substituting q for k in (5).

Just as there is an equation which relates densities and velocities to flow rates in a traffic stream there is also one which relates travel time to densities and flow rates. Both depend upon the principle of conservation of matter; while one can be derived from the other they do not always provide identical solutions.

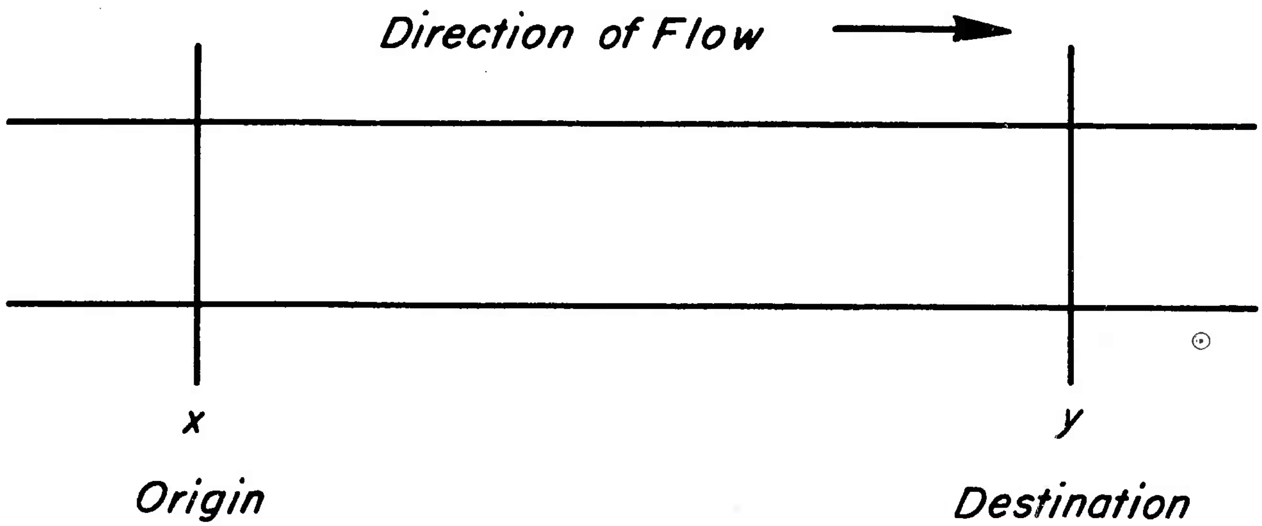

Fig. 2 - TRAVEL FROM x TO y

The time to travel a fixed distance in the traffic stream can be obtained as follows: Consider Figure 2 where two lines, x and y, are drawn across a road. At time t, an observer at y makes a count of the total number of vehicles in the interval (x,y). The traffic stream flows from x to y (i.e., from left to right); at a time $t + T(x,y,t)$ the vehicle located at x at time t passes the observer at y. $T(x,y,t)$ is the travel time of a vehicle going from x to y with its trip originating at the time t. Since the total count of vehicles along the road at time t must equal the total

flow past the observer in the interval $(t, t + T)$, we have

$$(6) \qquad \int_x^y k(r,t)dr = \int_t^{t+T(x,y,t)} q(y,s)ds \quad .$$

If the flow rate is a constant and if the density is only a function of x, i.e., stream velocity is not primarily constrained by stream density, the explicit solution of (6) is independent of the arrival time t and only depends upon the length of the trip.

The relation between the integral and the differential formulation of the conservation of fluid is easy to establish by taking the partial derivative of Equation (6) with respect to y and observing that the velocity of the stream at a point y and a time $t + T(x,y,t)$ equals the reciprocal of the partial of the travel time with respect to the trip destination, i.e.,

$$(7) \qquad v(y, t + T) = \left[\frac{\partial T}{\partial y}(x,y,t)\right]^{-1} \quad .$$

Since solutions of (5) can be expressed in terms of $u(k)$ as well as position and time, it should also be clear that Equation (6) can be rewritten in terms of the speed of shock waves.

2. The Motion of a Shock Wave

In this section we use Equation (4) to predict the motion of a shock wave which develops upstream from a bottleneck and influences the flow rate of the traffic stream before it arrives at the bottleneck. In the following section these solutions are used in Equation (6) to predict the travel time through the shock wave.

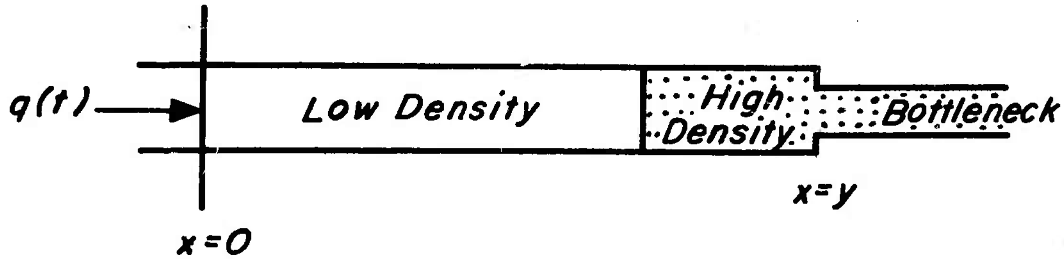

Fig.3- THE POSITION OF A SHOCK WAVE.

Consider the flow rate into the bottleneck of Figure 3. The beginning of the bottleneck is located at $x = y$; for our purposes $x = 0$ marks the origin of trips through the stream and also provides a convenient reference point where flow rates and densities can be measured. The stream moves from left to right; the flow rate at the origin is $q(0,t) = q(t)$ * and the density is $k(x,t) = k(t)$. At time zero we assume that the flow rate is low enough so that the constant capacity flow rate of the bottleneck is greater than the flow rate, $q(y,t)$. If we neglect statistical variations in the arrivals of vehicles at the bottleneck and if we assume that the flow rate into the bottleneck is less than its capacity, no queue of vehicles forms. As the flow rate increases with time and exceeds the capacity of the bottleneck, a queue forms and grows back into the stream. As the time-dependent flow rate peaks and begins to fall off, the queue will continue to increase until the flow rate into the queue is less than the capacity flow rate out of the bottleneck. The queue will vanish some time after peak flows have been observed at the origin.

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*Cumulative flows are denoted by capital letters, i.e., $Q(t) = \int_0^t q(t)dt$.

If the fluid travelled infinitely fast from the origin to the bottleneck, we would find that the size of the queue, $I(t)$, would equal cumulative net flows into the bottleneck.

(8) $$I(t) = Q(t) - Q(s_0) - C(t - s_0) \qquad s_0 \leq t$$

where s_0 is the instant when flow rates first exceed C . If we also assume that each vehicle occupied a space in the queue equal to the reciprocal of the jam density, k_j , the position of the shock wave would represent the head of the queue and would be located at

(9) $$x(t) = y - k_j^{-1} I(t) \qquad s_0 \leq t \quad .$$

The smallest value of $x(t)$ would occur at the time when the queue size has its largest value, i.e., that instant of time $t = s_1$ which is the smallest root of

(10) $$q(t) - C = 0 \quad .$$

On the other hand, if we include the effects of finite stream velocity, the flow rate at the bottleneck first equals the capacity, C , at a time $t_0 > s_0$ when

(11) $$q(y, t_0) = C \quad .$$

If we further assume that outside the queue the stream velocity is the free velocity, v_f , and is independent of position, the flow rate at a point x is

(12) $$q(x,t) = q(0, t - v_f^{-1}x) = q(t - v_f^{-1}x) \quad .$$

In other words, the flow rate function at x is equal to the flow rate function at the origin shifted by the time to travel from 0 to x. The solution of (11) becomes,

$$t_0 = s_0 + yv_f^{-1} \quad . \tag{13}$$

Consider that there are only two density regions: one where vehicles are outside the queue, the other where vehicles are closely packed in the queue that forms upstream from the bottleneck:

(14a) $\qquad k(x,t) = v_f^{-1} q(x,t) \qquad x$ outside the queue

(14b) $\qquad \qquad = k_j \qquad x$ inside the queue

Substituting (14) for k_1 and k_2 and $q(x,t)$ and C for q_1 and q_2 in (4), we obtain the velocity of the shock wave, in terms of its co-ordinate at time t.

$$u(t) = \frac{dx}{dt} = \frac{q(t - v_f^{-1}x) - C}{v_f^{-1} q(t - v_f^{-1}x) - k_j} \quad . \tag{15}$$

This first order differential equation is nonlinear in x and its derivative. While numerical solutions can be found, a useful analytic expression for the position of the shock wave as a function of time can be obtained by expanding the right-hand side of (15) about t and neglecting terms smaller than xv_f^{-2}. The result is a first order linear differential equation of the form,

$$\frac{dx}{dt} + a(t)x(t) = b(t) \tag{16}$$

where $a(t)$, $b(t)$ are time-varying coefficients involving the flow rate and its derivatives. The complete solution of (16) satisfying the initial condition that the shock wave be located at $x = y$ when the flow rate at the bottleneck first exceeds C is

$$x(t) = \frac{C(t - t_0) - (Q(t) - Q(t_0))}{k_j - k(t)} + \frac{k_j - k(t_0)}{k_j - k(t)} y \tag{17}$$

where $k(t) = k(0,t)$ and t_0 is given by (13). The shock wave reaches its extreme upstream point at time t_1 when

$$x(t_1) = (q(t_1) - C)v_f \left(\frac{dq}{dt}\right)^{-1} \tag{18}$$

and returns to the bottleneck at t_2 when $x(t_2) = y$. By substituting y for the left-hand side of (14) we find that t_2 is explicitly independent of the jam density,

$$Q(t_2) - Q(t_0) - C(t_2 - t_0) = y\left[k(t_2) - k(t_0)\right] \quad . \tag{19}$$

By neglecting terms smaller than yv_f^{-2} we find that this return time, t_2 , can also be obtained by equating the cumulative flows into the bottleneck with the maximum flow rates which can pass through the bottleneck in the capacity constrained interval (t_0 , t_2) , i.e.,

$$Q(t_2 - yv_f^{-1}) - Q(t_0 - yv_f^{-1}) = C(t_2 - t_0) \quad . \tag{20}$$

In other words the <u>duration</u> of the capacity constrained interval is not (as a first approximation) affected by the finite size of the vehicles or by the average speed of the vehicles within the queue itself even through the shock wave

influences upstream densities and flow rates during the course of its progress. If the density of vehicles at the origin at time t is small and close to the density at time t_0, the second term in (14) is approximately equal to y; the first term is approximately equal to $k_j^{-1}\left[Q(t) - Q(t_0) - C(t - t_0)\right]$ which, except for the factor k_j^{-1}, is identical to Equation (8).

Figure 4 is a plot of the position of the shock wave which forms upstream from the bottleneck in Figure 3. The flow rate at the origin is $q(t) = 3{,}000\ t^2 e^{-t}$ cars/hour where t is measured in hours. The bottleneck is five miles downstream from the origin, $C = 1450$ cars/hour and $k_j = 220$ cars/mile. The solid lines are plots of the shock wave for several values of the free velocity v_f and its distance from the origin in miles is shown on the right-hand ordinate; the dashed line is a plot of the flow rate $q(t)$ with values given on the left-hand ordinate.

As $v_f \longrightarrow \infty$ the position of the shock wave is given by Equation (9); it starts at $t_0 = 0.700$ hour and has its smallest value when $t = 1.00$ hour. As the velocity of the stream outside the queue is changed, three effects can be observed: (i) The shock wave starts later as v_f is reduced because it takes a longer time for vehicles to travel from the origin to the bottleneck, (ii) the largest upstream position of the shock wave increases as v_f gets smaller, (iii) the width of the peak narrows as v_f decreases.

3. Travel Time Through the Shock Wave

Consider the time to make a trip from the origin to the bottleneck in Figure 3. Part of the trip is made in the low-density region, part in the high-density region, i.e.,

$$\begin{aligned}(\text{total travel time}) &= (\text{time to travel to shock wave}) \\ &\quad + (\text{time to travel beyond shock wave})\ .\end{aligned}$$

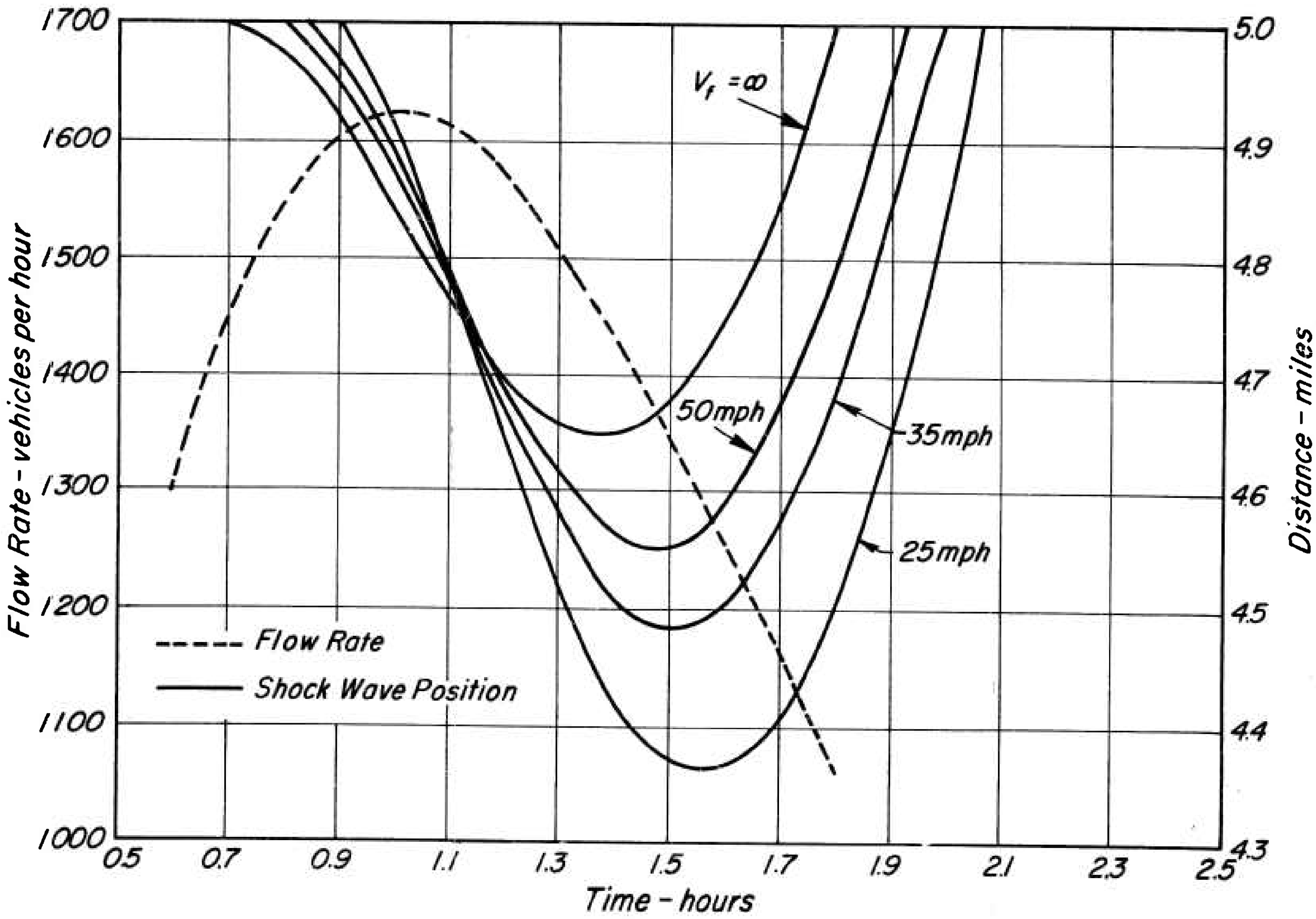

Fig.4-POSITION OF THE SHOCK WAVE.

Substituting (14) into (6) for $t > t_0$ we get

$$(21) \qquad T(0,y,t) = C^{-1}\left\{ \int_0^{x(t)} v_f^{-1} q(t - uv_f^{-1})du + \int_{x(t)}^{y} k_j du \right\}$$

$$= C^{-1}\left\{ Q(t) - Q(t - v_f^{-1}x(t)) + k_j(y - x(t)) \right\} .$$

Equation (21) can be written in two alternative forms. Expanding about t, neglecting terms $O(v_f^{-2})$ we can express the travel time in terms of the normalized density $\eta(t) = k(t)/k_j \leq 1$,

$$(22) \qquad T(0,y,t) = C^{-1}k_j \left[(y - x(t) + \eta(t)x(t)\right] \qquad t \geq t_0 .$$

In other words, the total travel time is the time to travel through a queue moving at the constant velocity Ck_j^{-1} plus a term which is proportional to the product of the nromalized density and the position of the shock wave at the origin of the trip.

From (21) we also have

$$(23) \qquad T(0,y,t) = C^{-1}y \quad \alpha(t)k(t) + (1 - \alpha(t))k_j$$

where $\alpha(t)$ is the fraction of the trip covered by the high-density or slow moving region at time t. Since $\alpha(t) \leq 1$, the term in brackets is a density lying between the density observed at the beginning of the trip and the jam density. Hence upper and lower bounds can be placed on the total travel time. As the velocity in the low-density region becomes very large, i.e., as $v_f \longrightarrow \infty$, we have

$$(24) \qquad T(0,y,t) \longrightarrow \frac{I(t)}{C} .$$

This simply states that the total travel time approaches the delay of a vehicle joining a queue of size $I(t)$ which is then processed at a constant service rate C .

We have used v_f to denote the ("free") speed of the traffic stream in the low-density region. The results can be generalized to include cases where the velocity in the low-density region is constant but less than the free speed. For example, the stream speed may be given by expressions of the form $v = v(k) < v_f$.

In generalizing these results at least one important effect would have to be considered. The first of these would be a more realistic assumption about the behavior of the stream density in the neighborhood of the beginning of the queue. Vehicles will not immediately slow down from v_f to Ck_j^{-1} on passing through the shock wave. Instead, they will slow down more gradually. Since the density will also change more gradually, the leading edge of the shock wave will give way to a region where the changes in flow with respect to density is large but not infinite. Even the simplest assumptions about vehicle deceleration characteristics seem to lead to much more complicated expressions than those we have already obtained; for this reason they are not presented here.

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